

The Analytic Classification of Plane Branches

Abramo Hefez¹ and Marcelo E. Hernandes²

Abstract

In this paper we give a solution to the open classical problem of analytic classification of plane branches.

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1. INTRODUCTION

The aim of this work is to present a solution to the problem of effective analytic classification of plane branches.

Oscar Zariski, in a course taught at the École Polytechnique [11], in 1973, inspired by the work of S. Ebey [4], exposed his research on the problem of analytic classification of plane branches belonging to a given equisingularity class. A great amount of work is dedicated there to the analysis of particular examples, showing the need of more effective methods to solve the problem.

To this respect, in the introduction of [11], Zariski wrote:

Le problème de la description complète de l'espace des modules M d'une classe d'équisingularité donnée est entièrement ouvert et les quelques exemples du chapitre V montrent que M a une structure trop complexe pour espérer répondre totalement à la question.

La question, pourtant plus restrictive, de la détermination de la dimension de la "composante générique" de M n'est pas résolue.

The first non-trivial result in this direction was given by C. Delorme in [3], where he answered the above second question in a very particular case, describing the generic component of the moduli space for plane branches with one Puiseux pair and computing its dimension.

In this paper, we show how one can break the complexity of the moduli space by stratifying the given equisingularity class by means of a *good numerical invariant* that separates branches into finitely many types, such that analytic equivalence in each stratum is manageable.

This is accomplished by considering the sets Λ of values of Kähler differentials on branches as finer numerical invariants than the semigroup of values Γ which characterizes the equisingularity class. If one stratifies with

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such sets Λ the open set of the affine space representing the parameter space of an equisingularity class, corresponding to Puiseux parametrizations with fixed Puiseux pairs, and pick a set of special representatives in a specified normal form for each stratum, then the group action that represents analytic equivalence becomes rather trivial, allowing us to solve in general the above problems posed by Zariski.

Our setup is similar to that of [11], adding to it two techniques with computational flavor. The first one is the use of a *SAGBI* algorithm, due to L. Robbiano and M. Sweedler [10], that we adapted to our situation in [7] and [9], to describe privileged bases of the local rings of plane branches as well of the modules of Kähler differentials of these local rings, allowing us to compute the set of invariants Λ . The second technique is the algorithm of *Complete Transversal* due to J. W. Bruce, N. P. Kirk and A. A. du Plessis [2] that allows to determine all normal forms of map-germs under one of Mather's group action, but doesn't allow to predict a priori what will be the outgoing result. The strength of our method stems in the conjugation of these two tools that allows, via the existence of some differentials, to control each step of the Complete Transversal algorithm, giving explicitly all possible normal forms and conditions for the analytic equivalence of germs in normal forms.

All the results we obtain are effective, in the sense that there is an efficient algorithm that puts any plane branch into its normal form and it is easy to recognize whether two plane branches under normal form are equivalent or not. The whole process has been implemented³.

2. PRELIMINARIES

Our ground field is the field \mathbb{C} of complex numbers. Since all power series we will work with are finitely determined, with respect to the equivalence relations we will consider, the results in this work are valid in the formal context and in the analytic context, as well. In order to have a geometric interpretation of our objects, we will adopt the analytic point of view. The reader who desires to find all the known results quoted in this section is invited to consult [6], where they are gathered with their proofs.

We denote by \mathcal{O}_2 the ring $\mathbb{C}\{X, Y\}$ of convergent power series in two variables with coefficients in \mathbb{C} . Let f be in the maximal ideal \mathcal{M}_2 of \mathcal{O}_2 and irreducible. Then the class of f in \mathcal{O}_2 , modulo associates, is called a *plane branch* and denoted by (f) . We will identify (f) with the germ of analytic plane curve at the origin $\{(x, y) \in (\mathbb{C}^2, 0); f(x, y) = 0\}$.

We say that two plane branches (f_1) and (f_2) are *equisingular*, writing $(f_1) \equiv (f_2)$ if and only if (f_1) and (f_2) are topologically equivalent as complex immersed germs of curves in $(\mathbb{C}^2, 0)$; that is, when there exists a homeomorphism $\Phi : U \rightarrow U'$, where U and U' are neighborhoods of

³cf. www.dma.uem.br/~hernandes/publications.html

the origin in \mathbb{C}^2 such that f_1 (resp. f_2) is convergent in U (resp. U') and $\Phi((f_1) \cap U) = (f_2) \cap U'$. The set of all plane branches which are equisingular to each other will be called an *equisingularity class*.

When the above transformation Φ is an analytic isomorphism, we say that (f_1) and (f_2) are *analytically equivalent*, or shortly *equivalent*, writing in this case $(f_1) \sim (f_2)$. So, two branches (f_1) and (f_2) are equivalent if, and only if, there are an automorphism Φ^* and a unit u , both of \mathcal{O}_2 , such that $\Phi^*(f_1) = uf_2$.

If we denote by \mathcal{O}_f the quotient ring $\mathcal{O}_2/\langle f \rangle$, where $\langle f \rangle$ represents the ideal generated by f , then one has $(f_1) \sim (f_2)$ if, and only if, $\mathcal{O}_{f_1} \simeq \mathcal{O}_{f_2}$, as \mathbb{C} -algebras.

Our main concern in this work is to perform the analytic classification of plane branches within a given equisingularity class.

Since the integral closure of \mathcal{O}_f in its field of fractions is a complete discrete valuation ring, hence isomorphic to $\mathcal{O}_1 = \mathbb{C}\{t\}$, any plane branch f has a parametrization $\varphi(t) = (x(t), y(t))$ with $x(t)$ and $y(t)$ in \mathcal{M}_1 , the maximal ideal of \mathcal{O}_1 , not both identically zero, such that $f(x(t), y(t)) = 0$. Conversely, any non-zero mapping $\varphi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$, also called a *parametrization*, determines a plane branch. We will call a parametrization *primitive* if it cannot be reparametrized by a power of a new variable.

It is a well known fact, already used in [11] (see also Lemma 2.2 in [1]) that, given two plane branches (f_1) and (f_2) , parametrized, respectively, by φ_1 and φ_2 , then $(f_1) \sim (f_2)$ if, and only if, φ_1 and φ_2 are \mathcal{A} -equivalent, writing $\varphi_1 \sim_{\mathcal{A}} \varphi_2$, where \mathcal{A} -equivalence means that there exist germs of analytic isomorphisms σ and ρ of $(\mathbb{C}^2, 0)$ and $(\mathbb{C}, 0)$, respectively, such that $\varphi_2 = \sigma \circ \varphi_1 \circ \rho^{-1}$.

So, the analytic classification of plane branches reduces to the \mathcal{A} -classification of parametrizations, which we are going to undertake in this paper.

Any plane branch is known (cf. [11]) to be equivalent to a branch with a (primitive) Puiseux parametrization:

$$(2.1) \quad \varphi(t) = (x(t), y(t)) = (t^{\beta_0}, \sum_{i \geq \beta_1} a_i t^i),$$

with $a_i \in \mathbb{C}$, $a_{\beta_1} = 1$, $\beta_0 < \beta_1$, $\beta_0 \nmid \beta_1$ and $f \circ \varphi(t) = 0$.

The *characteristic exponents* of φ are the integers β_0, β_1, \dots , where for $i \geq 1$,

$$\beta_i = \min\{j; a_j \neq 0 \text{ and } \text{GCD}(\beta_0, \dots, \beta_{i-1}, j) \neq \text{GCD}(\beta_0, \dots, \beta_{i-1})\}.$$

The associated integers n_i are $n_0 = 1$ and

$$n_i = \frac{e_{i-1}}{e_i},$$

where $e_0 = \beta_0$ and $e_i = \text{GCD}(\beta_0, \dots, \beta_i)$.

Since the parametrization is primitive, there is a positive integer g such that $e_g = 1$. This g is called the *genus* of the branch. The Puiseux pairs of φ are (n_i, m_i) , $i = 1, \dots, g$, where $m_i = \frac{\beta_i}{e_i}$.

It is a classical result, going back to the thirties, due essentially to Brauner and Zariski, that the Puiseux pairs form a complete set of numerical invariants for the topological classification of plane branches; that is, if f_1 and f_2 have parametrizations as (2.1), then $(f_1) \equiv (f_2)$ if, and only if, (f_1) and (f_2) have the same Puiseux pairs or, equivalently, the same characteristic exponents.

So, the Puiseux parametrizations as (2.1) with $a_{\beta_i} \neq 0$, for $i = 2, \dots, g$, and such that $a_j = 0$, if $\beta_i \leq j < \beta_{i+1}$ and $e_i \nmid j$, determine equisingular branches; and any plane branch in this equisingularity class is equivalent to one with such a Puiseux parametrization.

Any parametrization φ induces a homomorphism

$$\begin{aligned} \varphi^*: \mathcal{O}_2 &\rightarrow \mathcal{O}_1. \\ h &\mapsto h \circ \varphi(t) \end{aligned}$$

Assuming φ primitive, we define the value $v_\varphi(h)$, for $h \in \mathcal{O}_2$, as being $\text{ord}_t(\varphi^*(h))$, the order in t of the power series $\varphi^*(h)$, defining also $v_\varphi(0) = \infty$. The semigroup of the extended naturals $\Gamma_\varphi = v_\varphi(\mathcal{O}_2)$ will be called the *semigroup of values* of φ . This semigroup will be represented by $\Gamma_\varphi = \langle v_0, v_1, \dots, v_g \rangle$, where $v_0 < \dots < v_g$ is its minimal set of generators. When φ is a Puiseux parametrization, it is well known (cf. [11]) that there are relations among these v_i 's and the β_i 's, given by $\beta_0 = v_0$, $\beta_1 = v_1$, $\text{GCD}(v_0, \dots, v_i) = e_i$, and

$$v_{i+1} = n_i v_i + \beta_{i+1} - \beta_i.$$

Hence, the characteristic exponents of φ and Γ_φ determine each other.

The semigroup Γ_φ has a conductor; that is, there is a natural number c such that $c - 1 \notin \Gamma_\varphi$, and $l \in \Gamma_\varphi$, for all $l \geq c$. This means that the set $\mathbb{N} \setminus \Gamma_\varphi$, whose elements are called the *gaps* of Γ_φ , is finite.

So, in order to preserve the Puiseux form given in (2.1), under the \mathcal{A} -action, the analytic isomorphisms σ and ρ must be of a very special type, as described below.

If $\varphi(t) = (x(t), y(t))$ and $\varphi_1(t_1) = (x_1(t_1), y_1(t_1))$ are two parametrizations as in (2.1), then in order to have $\varphi_1(t_1) = \sigma \circ \varphi \circ \rho^{-1}(t_1)$, it is necessary and sufficient that

$$\begin{aligned} \sigma(X, Y) &= (r^{v_0} X + p, r^{v_1} Y + q), \\ (2.2) \quad t_1 &= \rho(t) = r t^{\vartheta} \sqrt[{\vartheta}]{1 + \frac{\varphi^*(p)}{r^{v_0} t^{v_0}}}, \end{aligned}$$

where $r \in \mathbb{C}^*$ and $p, q \in \mathcal{O}_2$, with $v_\varphi(p) > v_0$ and $v_\varphi(q) > v_1$.

The \mathcal{A} -action induced on parametrizations of the form (2.1) is then given by $(t^{v_0}, y(t)) \sim_{\mathcal{A}} (t_1^{v_0}, y_1(t_1))$ if, and only if,

$$(2.3) \quad y_1(t_1) = r^{v_1} y(\rho^{-1}(t_1)) + q(\rho^{-1}(t_1)^{v_0}, y(\rho^{-1}(t_1))).$$

Ebey and Zariski (cf. [4] and [11]) gave some elimination criteria **(EC)** of parameters of $y(t)$ in a given parametrization $(t^{v_0}, y(t))$ as in (2.1), by means of the \mathcal{A} -equivalence.

Let $\varphi = (t^{v_0}, t^{v_1} + \sum_{i>v_1} a_i t^i)$ be a Puiseux parametrization, and let $j > v_1$ be an integer. If one of the following conditions holds,

EC1) $j \in \Gamma_{\varphi}$, or

EC2) $j + v_0 - v_1 \in \Gamma_{\varphi}$,

then φ is \mathcal{A} -equivalent to a parametrization $(t^{v_0}, t^{v_1} + \sum_{i>v_1} a'_i t^i)$, with $a'_i = a_i$, when $i < j$, and $a'_j = 0$.

It then follows that any parametrization $\varphi = (t^{v_0}, \sum_{v_1 \leq i} a_i t^i)$ is \mathcal{A} -equivalent to the parametrization

$$(t^{v_0}, \sum_{v_1 \leq i < c} a_i t^i),$$

where c is the conductor of Γ_{φ} .

Let Σ_{Γ} denote the set of all parametrizations of the above form, such that Γ_{φ} is equal to a given Γ . This set can be identified with an open set (the complement of the union of the hyperplanes $a_{\beta_i} = 0$, $i = 2, \dots, g$) of an affine space, whose points are the ordered sets of the coefficients of $y(t)$ which are not necessarily zero, excluding a_{β_1} , which is taken to be 1. So, now, we are reduced to classify, modulo \mathcal{A} -equivalence, the parametrizations in the set Σ_{Γ} , in order to classify analytically plane branches.

Zariski noticed in [11] that, in order to get more Elimination Criteria then the above ones, it was necessary to consider the module of Kähler differentials over the local ring of the branch, introducing new numerical analytic invariants.

Let us denote by

$$\Omega_2 = \{hdX + gdY; \ g, h \in \mathcal{O}_2\},$$

the \mathcal{O}_2 -free module of germs of differentials at $(\mathbb{C}^2, 0)$, and by

$$\Omega_1 = \{\xi dt; \ \xi \in \mathcal{O}_1\},$$

the \mathcal{O}_1 -free module of germs of differentials at $(\mathbb{C}, 0)$.

A parametrization $\varphi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$, $t \mapsto (x(t), y(t))$, induces a natural \mathcal{O}_2 -modules homomorphism (thought as an extension of the map $\varphi^* : \mathcal{O}_2 \rightarrow \mathcal{O}_1$):

$$\begin{array}{ccc} \varphi^* : & \Omega_2 & \longrightarrow \Omega_1. \\ & hdX + gdY & \longmapsto (\varphi^*(h)x'(t) + \varphi^*(g)y'(t))dt \end{array}$$

The image of Ω_2 in Ω_1 under φ^* is isomorphic to the quotient of the module of Kähler differentials over the local ring $(\varphi^*(\mathcal{O}_2))$ of the branch determined by φ , by its torsion submodule (cf. [7] or [9]).

A primitive parametrization φ also induces a valuation v_φ on Ω_2 , defined by

$$v_\varphi(hdX + gdY) = \text{ord}_t(\varphi^*(h)x'(t) + \varphi^*(g)y'(t)) + 1.$$

We now define

$$\Lambda_\varphi = v_\varphi(\Omega_2).$$

The set Λ_φ is \mathcal{A} -invariant, as we will show later, and will play a key role in our solution of the classification problem.

Since for all $h \in \mathcal{M}_2$ we have that $v_\varphi(dh) = v_\varphi(h)$, it follows that $\Gamma_\varphi \setminus \{0\} \subset \Lambda_\varphi$; and because Γ_φ has a conductor, we have that the set $\Lambda_\varphi \setminus \Gamma_\varphi$, as a subset of the set of gaps of Γ_φ , is finite.

Zariski in [12] has shown that $\Lambda_\varphi \setminus \Gamma_\varphi = \emptyset$ if, and only if, φ is \mathcal{A} -equivalent to the parametrization (t^{v_0}, t^{v_1}) , where $\text{GCD}(v_0, v_1) = 1$.

Since Λ_φ and Γ_φ are \mathcal{A} -invariant, it follows that, if $\Lambda_\varphi \setminus \Gamma_\varphi \neq \emptyset$, then

$$\lambda = \min(\Lambda_\varphi \setminus \Gamma_\varphi) - v_0,$$

is an invariant under the \mathcal{A} -action, called the Zariski invariant of φ . It is known (see [11]) that such a φ is \mathcal{A} -equivalent to a Puiseux parametrization of the form

$$(2.4) \quad \varphi = (t^{v_0}, t^{v_1} + t^\lambda + \sum_{\lambda < i < c} a_i t^i),$$

and, in this case,

$$\lambda = v_\varphi(v_0 X dY - v_1 Y dX) - v_0.$$

Related to the invariant λ , Zariski in [11] proved the following extra elimination criterion:

EC3) If φ is as in (2.4) and $j - \lambda$ is in the semigroup generated by v_0 and v_1 , then φ is \mathcal{A} -equivalent to a parametrization $(t^{v_0}, t^{v_1} + t^\lambda + \sum_{\lambda < i < c} a'_i t^i)$, with $a'_i = a_i$, when $i < j$, and $a'_j = 0$.

The above criterion doesn't work for all the equisingularity class, but depends upon the \mathcal{A} -equivalence class of the parametrization φ .

In the next theorem, our central result in this work, we will determine all possible such elimination criteria, which will lead us to what we call the *normal forms* for the Puiseux parametrizations.

Theorem 2.1 (The Normal Forms Theorem). *Let $\varphi \in \Sigma_\Gamma$ be a Puiseux parametrization of a plane branch with semigroup of values $\Gamma = \langle v_0, v_1, \dots, v_g \rangle$. Then, either φ is \mathcal{A} -equivalent to the monomial parametrization (t^{v_0}, t^{v_1}) , or it is \mathcal{A} -equivalent to a parametrization*

$$(2.5) \quad (t^{v_0}, t^{v_1} + t^\lambda + \sum_{\substack{i > \lambda \\ i \notin \Lambda - v_0}} a_i t^i),$$

where λ is its Zariski invariant and $\Lambda = \Lambda_\varphi$ is the set of orders of differentials of the branch. Moreover, if φ and φ' (with coefficients a'_i instead of a_i) are parametrizations as in (2.5), representing two plane branches with same

semigroup of values and same set of values of differentials, then $\varphi \sim_{\mathcal{A}} \varphi'$ if and only if there is $r \in \mathbb{C}^*$ such that $r^{\lambda-v_1} = 1$ and $a_i = r^{i-v_1} a'_i$, for all i .

Remark that the \mathcal{A} -normal form in (2.5) is completely determined by the semigroup Γ and the set Λ . So, once Γ is fixed, the number of \mathcal{A} -normal forms is finite, corresponding to all possible sets Λ in the equisingularity class determined by Γ , which may be computed by the algorithm presented in [9].

The above theorem gives the ultimate elimination criterion **EC** that contains all the known criteria **EC1**, **EC2** and **EC3**:

EC) If φ is as in (2.4) and $j + v_0 \in \Lambda_\varphi$, $j > \lambda$, then φ is \mathcal{A} -equivalent to a parametrization $(t^{v_0}, t^{v_1} + t^\lambda + \sum_{\lambda < i < c} a'_i t^i)$, with $a'_i = a_i$, when $i < j$, and $a'_j = 0$.

The rest of the paper is devoted to prove Theorem 2.1.

3. ORBITS AND THEIR TANGENT SPACES

We will assume the reader familiar with the language of singularity theory. We will denote by $j^k(h)$ the k -jet of an element h .

Let $\text{Aut}(\mathbb{C}^n, 0)$ denote the group of germs of analytic automorphisms of $(\mathbb{C}^n, 0)$, and let $\text{Aut}_1(\mathbb{C}^n, 0)$ the subgroup of elements $A \in \text{Aut}(\mathbb{C}^n, 0)$ such that $j^1(A) = \text{Id}$.

We also denote by $\widetilde{\text{Aut}}(\mathbb{C}^2, 0)$ the subgroup of elements A of $\text{Aut}(\mathbb{C}^2, 0)$ such that $j^1(A) = (X + \beta Y, Y)$, with $\beta \in \mathbb{C}$.

We say that the Puiseux parametrizations φ_1 and φ_2 are \mathcal{A}_1 -equivalent (resp. $\widetilde{\mathcal{A}}$ -equivalent) if $\varphi_2 = \sigma \circ \varphi_1 \circ \rho^{-1}$ with $\sigma \in \text{Aut}_1(\mathbb{C}^2, 0)$ (resp. $\sigma \in \widetilde{\text{Aut}}(\mathbb{C}^2, 0)$) and $\rho \in \text{Aut}_1(\mathbb{C}, 0)$.

We say that φ_1 and φ_2 in Σ_Γ are *homothetic*, or \mathcal{H} -equivalent if $\varphi_2 = \sigma \circ \varphi_1 \circ \rho^{-1}$, with $\rho(t) = \alpha t$ and $\sigma(X, Y) = (\alpha^{v_0} X, \alpha^{v_1} Y)$, for some $\alpha \in \mathbb{C}^*$.

So, the \mathcal{A} -action on the space of Puiseux series representing an equisingularity class may be obtained by the $\widetilde{\mathcal{A}}$ -action followed by the \mathcal{H} -action.

If \mathcal{G} represents one of the actions \mathcal{A} , \mathcal{A}_1 or $\widetilde{\mathcal{A}}$, then \mathcal{G}^k will represent the Lie group action of k -jets of corresponding automorphisms on the space Σ_Γ^k of k -jets of elements of Σ_Γ .

Let us recall (a special case) of the Complete Transversal Theorem of [2], adapted to our use:

The Complete Transversal Theorem. *Let G be a Lie group acting smoothly on an open set U of an affine space A with underlying vector space V , and let W be a subspace of V such that $\forall g \in G, \forall v \in U$ and $\forall w \in W$ with $v + w \in U$ and $g \cdot v + w \in U$, one has*

$$g \cdot (v + w) = g \cdot v + w.$$

If $W \subset T_v(G \cdot v)$, with $v \in U$, and $T_v(G \cdot v)$ is the tangent space at v to the orbit $G \cdot v$, then for every $w \in W$ such that $v + w \in U$, one has

$$G(v + w) = G \cdot v.$$

Although we are mainly interested in the \mathcal{A} -equivalence, we will start analyzing the unipotent \mathcal{A}_1 -action, passing to the $\tilde{\mathcal{A}}$ -action and, finally, applying homotheties, to get to the \mathcal{A} -equivalence.

Let U be the open set Σ_{Γ}^k of the appropriate affine space and let $G = \mathcal{A}_1^k$. Then the initial hypothesis at the beginning of the Complete Transversal Theorem is fulfilled for $W = \{(0, bt^k); b \in \mathbb{C}\}$.

The tangent spaces to the orbits $\mathcal{A}_1^k \cdot \varphi$ and $\tilde{\mathcal{A}}^k \cdot \varphi$ at an element $\varphi = (x(t), y(t)) \in \Sigma_{\Gamma}^k$ are given by:

$$(3.1) \quad T_{\varphi}(\mathcal{A}_1^k \cdot \varphi) = \left\{ j^k((x'(t), y'(t))\epsilon + (\varphi^*(g), \varphi^*(h))); \epsilon \in \mathcal{M}_1^2, g, h \in \mathcal{M}_2^2 \right\},$$

and

$$(3.2) \quad T_{\varphi}(\tilde{\mathcal{A}}^k \cdot \varphi) = \left\{ j^k((x'(t), y'(t))\epsilon + (\varphi^*(g), \varphi^*(h))); \epsilon \in \mathcal{M}_1^2, h \in \mathcal{M}_2^2, \right.$$

$$\left. g \in \langle X^2, Y \rangle \right\}.$$

The proof of (3.1) may be found, for example, in [5], while the proof of (3.2) may be obtained in a similar way.

We will show how, by using the Complete Transversal Theorem, one obtains all normal forms of Puiseux parametrizations, with respect to the \mathcal{A}_1 -equivalence, by eliminating terms in the expansion of $y(t)$, finding more elimination criteria than the general ones of Ebey and Zariski, adapted to a specific branch. The idea is to verify at each step if the k -jet of the parametrization is \mathcal{A}_1^k -equivalent to its $(k-1)$ -jet, which implies that the term of degree k in $y(t)$ can be eliminated under the \mathcal{A}_1 -action. For this, according to the Complete Transversal Theorem, it is enough to verify if the vector $(0, bt^k)$ belongs to the tangent space to the \mathcal{A}_1^k -orbit of the k -jet of the parametrization, and this fact may be expressed in terms of the existence of differentials in $(\mathbb{C}^2, 0)$ of certain order with respect to the valuation determined by the parametrization, as we will see soon. The procedure will stop after finitely many steps since all terms in $y(t)$ of order greater or equal to the conductor c of the semigroup of values of the branch are eliminable. Next, we will find the normal forms under the $\tilde{\mathcal{A}}$ -action by analyzing separately some few remaining cases. Finally, the normal forms under the \mathcal{A} -action are obtained applying homotheties.

In order to apply this procedure, we will need to describe more explicitly the tangent spaces to orbits in Σ_{Γ}^k .

Lemma 3.1. *Let $k > v_1$ and $\varphi \in \Sigma_{\Gamma}^k$. For $b \neq 0$, we have that the vector $(0, bt^k)$ belongs to $T_{\varphi}(\mathcal{A}_1^k \cdot \varphi)$ (resp. to $T_{\varphi}(\tilde{\mathcal{A}}^k \cdot \varphi)$), if and only if there exist $g, h \in \mathcal{M}_2^2$ (resp. $g \in \langle X^2, Y \rangle, h \in \mathcal{M}_2^2$) such that*

$$(3.3) \quad k + v_0 - 1 = \text{ord}_t(\varphi^*(h)x'(t) - \varphi^*(g)y'(t)).$$

PROOF: We prove the result for $T_{\varphi}(\mathcal{A}_1^k \cdot \varphi)$, since the other situation is similar.

In order to have $(0, bt^k) \in T_\varphi(\mathcal{A}_1^k \cdot \varphi)$ it is necessary and sufficient to be able to solve the system:

$$\begin{cases} x'(t) \cdot \epsilon + \varphi^*(g) = 0 & \text{mod } t^{k+1} \\ y'(t) \cdot \epsilon + \varphi^*(h) = bt^k & \text{mod } t^{k+1}. \end{cases}$$

That is, $\epsilon = -\frac{\varphi^*(g)}{x'(t)} \text{ mod } t^{k+1}$. Notice that $\epsilon \in \mathcal{M}_1^2$, since $g \in \mathcal{M}_2^2$.

In this way we get the equation

$$bt^k = \frac{\varphi^*(h)x'(t) - \varphi^*(g)y'(t)}{x'(t)} \text{ mod } t^{k+1}.$$

So, $(0, bt^k) \in T_\varphi(\mathcal{A}_1^k \cdot \varphi)$ if, and only if, there exist $g, h \in \mathcal{M}_2^2$ satisfying (3.3). \blacksquare

For $i \in \mathbb{N}$, we define

$$\Omega_2^{(i)} = \{hdX + gdY \in \Omega_2; \ g, h \in \mathcal{M}_2^i\},$$

where we put $\mathcal{M}_2^0 = \mathcal{O}_2$. So, $\Omega_2^{(0)} = \Omega_2$.

Given a primitive parametrization φ , we also define

$$\Lambda_\varphi^i = v_\varphi(\Omega_2^{(i)}).$$

Notice that $\Lambda_\varphi^0 = \Lambda_\varphi$. These sets are invariant under \mathcal{A} -equivalence, as we show below.

Proposition 3.2. *If φ and φ_1 are \mathcal{A} -equivalent primitive parametrizations, then, for all $i \in \mathbb{N}$, we have $\Lambda_\varphi^i = \Lambda_{\varphi_1}^i$.*

PROOF: The commutative diagram,

$$\begin{array}{ccc} \mathbb{C}, 0 & \xrightarrow{\varphi} & \mathbb{C}^2, 0 \\ \downarrow \rho & & \downarrow \sigma \\ \mathbb{C}, 0 & \xrightarrow{\varphi_1} & \mathbb{C}^2, 0 \end{array}$$

with isomorphisms σ and ρ induces the following two diagrams:

$$\begin{array}{ccc} \mathcal{O}_2 & \xrightarrow{\varphi^*} & \mathcal{O}_1 \\ \uparrow \sigma^* & & \uparrow \rho^* \\ \mathcal{O}_2 & \xrightarrow{\varphi_1^*} & \mathcal{O}_1 \end{array} \quad \begin{array}{ccc} \Omega_2 & \xrightarrow{\varphi^*} & \Omega_1 \\ \uparrow \sigma^* & & \uparrow \rho^* \\ \Omega_2 & \xrightarrow{\varphi_1^*} & \Omega_1 \end{array}$$

with isomorphisms σ^* and ρ^* . Now the result follows by functoriality, observing that $\sigma^*(\mathcal{M}_2^i) = \mathcal{M}_2^i$. \blacksquare

If we define

$$\Omega'_2 = \{hdX + gdY \in \Omega_2; \ g \in \langle X^2, Y \rangle, h \in \mathcal{M}_2^2\},$$

and $\Lambda'_\varphi = v_\varphi(\Omega'_2)$, then Lemma 3.1 may be rephrased as follows:

Proposition 3.3. *Let $k > v_1$ and $\varphi \in \Sigma_\Gamma^k$. For $b \neq 0$, we have that $(0, bt^k)$ belongs to $T_\varphi(\mathcal{A}_1^k \cdot \varphi)$ (resp. to $T_\varphi(\tilde{\mathcal{A}}^k \cdot \varphi)$) if, and only if,*

$$(3.4) \quad k + v_0 \in \Lambda_\varphi^2 \quad (\text{resp. } k + v_0 \in \Lambda'_\varphi)$$

4. NORMAL \mathcal{A}_1 -FORMS

In this section we will find the normal forms of Puiseux parametrizations in an equisingularity class with given semigroup of values Γ , under \mathcal{A}_1 -equivalence. We begin with a proposition that will give us the recursion step.

Proposition 4.1. *Let $\varphi = (t^{v_0}, t^{v_1} + \sum_{v_1 < i < c} a_i t^i) \in \Sigma_\Gamma$ and let k be an integer such that $k + v_0 \in \Lambda_\varphi^2$. Then there exists $\varphi_1 \in \Sigma_\Gamma$ such that $\varphi_1 \sim_{\mathcal{A}_1} \varphi$ and*

$$j^k(\varphi_1) = j^{k-1}(\varphi_1) = j^{k-1}(\varphi).$$

PROOF: From Proposition 3.3, we have that the vector $(0, -a_k t^k)$ belongs to $T_{j^k(\varphi)}(\mathcal{A}_1^k \cdot j^k(\varphi))$, and therefore by the Complete Transversal Theorem it follows that $j^k(\varphi) \sim_{\mathcal{A}_1^k} j^{k-1}(\varphi)$. Hence, there are appropriate germs of analytic isomorphisms σ and ρ such that $\sigma \circ j^k(\varphi) \circ \rho^{-1} = j^{k-1}(\varphi)$. So, $j^k(\sigma \circ \varphi \circ \rho^{-1}) = j^{k-1}(\varphi)$. Now the result follows putting $\varphi_1 = \sigma \circ \varphi \circ \rho^{-1}$ ■

The following result will be important in the sequel.

Proposition 4.2. *Let $\varphi = (t^{v_0}, t^{v_1} + t^\lambda + \dots)$ be a Puiseux parametrization with $\Gamma_\varphi = \langle v_0, v_1, \dots, v_g \rangle$. If $S = \{v_0, 2v_0, v_1, v_0 + v_1, 2v_1, v_0 + \lambda\}$, then one has*

$$S \subseteq \Lambda_\varphi \setminus \Lambda_\varphi^2 \subseteq S \cup \{v_1 + \lambda\},$$

with equality on the top if, and only if, $n_1 = 2$ and $g \geq 2$.

PROOF: We have that $n \in \Lambda_\varphi \setminus \Lambda_\varphi^2$ if, and only if, $n = v_\varphi(\omega)$, where $\omega = h dX + g dY$, with $h \notin \mathcal{M}_2^2$ or $g \notin \mathcal{M}_2^2$.

We have that $S \subseteq \Lambda_\varphi \setminus \Lambda_\varphi^2$, since $v_\varphi(dX) = v_0$, $v_\varphi(dY) = v_1$, $v_\varphi(X dX) = 2v_0$, $v_\varphi(X dY) = v_0 + v_1$, $v_\varphi(Y dY) = 2v_1$, and $v_\varphi(v_1 Y dX - v_0 X dY) = v_0 + \lambda$.

Now, suppose that $v_\varphi(h dX + g dY) \notin S$, where $h = \alpha X + \beta Y + h_2$ and $g = aX + bY + g_2$, with $h_2, g_2 \in \mathcal{M}_2^2$, and one of the numbers a, b, α or β is not zero.

So, in this case, we must have $v_\varphi(h dX + g dY) > v_\varphi(h dX) = v_\varphi(g dY)$. This implies that

$$v_\varphi(h) + v_0 = v_\varphi(g) + v_1,$$

with $v_\varphi(h) = v_0$ or $v_\varphi(h) = v_1$ or $v_\varphi(g) = v_0$ or $v_\varphi(g) = v_1$.

If $v_\varphi(h) = v_0$, then we would have $2v_0 = v_\varphi(g) + v_1$, which is not possible. Hence $\alpha = 0$.

If $v_\varphi(g) = v_0$, then $v_\varphi(h) + v_0 = v_1 + v_0$, hence $v_\varphi(h) = v_1$.

If $v_\varphi(h) = v_1$, then $v_1 + v_0 = v_\varphi(g) + v_1$, so $v_\varphi(g) = v_0$. Hence, in this case, $v_\varphi(hdX + gdY) = v_0 + \lambda \in S$, which is to be excluded. Hence, $a = \beta = 0$.

So, the only remaining possibility is that $v_\varphi(g) = v_1$, in which case, $b \neq 0$ and $a = \alpha = \beta = 0$. So, we have $v_\varphi(h) + v_0 = 2v_1$, hence $v_1 < v_\varphi(h) < 2v_1$, which implies that $v_\varphi(h) = sv_1 + rv_0$, with $s = 0, 1$. We have that $s = 0$, because, otherwise, we would have $v_1 + (r + 1)v_0 = 2v_1$, which would imply that v_0 divides v_1 , a contradiction.

If the genus of φ is 1, then $v_0 = 2$, and in this case, because of **EC1** we have that φ is \mathcal{A} -equivalent to the parametrization (t^2, t^{v_1}) , hence not satisfying the hypothesis of the Proposition.

Therefore, $g \geq 2$, and $v_\varphi(h) = rv_0$. Therefore,

$$(r + 1)v_0 = 2v_1,$$

which implies $n_1 = 2$. Also, $v_\varphi(hdX + gdY) > 2v_1$, which in view of the expression of φ and the above equality implies that $v_\varphi(hdX + gdY) = v_1 + \lambda$.

Conversely, if $g \geq 2$ and $n_1 = 2$, we have that

$$v_\varphi(v_1X^r dX - v_0Y dY) = v_1 + \lambda,$$

where $(r + 1)v_0 = 2v_1$. ■

Now, we have the following result:

Proposition 4.3. *Let $\varphi \in \Sigma_\Gamma$ and set $\Lambda = \Lambda_\varphi$. Suppose that $\Lambda \setminus \Gamma \neq \emptyset$, and let λ be the Zariski invariant of φ . Then φ is \mathcal{A}_1 -equivalent to a parametrization*

$$(t^{v_0}, t^{v_1} + t^\lambda + \sum_{\substack{i \notin \Lambda^2 - v_0 \\ i > \lambda}} a_i t^i).$$

PROOF: First observe that since $\Lambda \setminus \Gamma \neq \emptyset$, it follows that $v_0 \geq 3$, so, in this situation, any integer $l + v_0$, where l is greater or equal than the conductor c of Γ , belongs to Λ but, by Proposition 4.2, it is not in $\Lambda \setminus \Lambda^2$, so it is in Λ^2 . This shows that the set $\mathbb{N} \setminus (\Lambda^2 - v_0)$ is finite (bounded by above by $c - 1$).

Let $\lambda_1, \dots, \lambda_s$ be the elements in $\Lambda^2 - v_0$ in the interval (λ, c) . From Proposition 4.1, there exists a Puiseux parametrization φ_1 with $\varphi_1 \sim_{\mathcal{A}_1} \varphi$ such that

$$j^{\lambda_1}(\varphi_1) = j^{\lambda_1-1}(\varphi_1) = j^{\lambda_1-1}(\varphi).$$

Next, do the same with φ_1 instead of φ and λ_2 instead of λ_1 , observing, by Proposition 3.2, that $\Lambda_{\varphi_1}^2 = \Lambda^2$; etc. ■

The next step will be to pass from the \mathcal{A}_1 -equivalence to the $\tilde{\mathcal{A}}$ -equivalence.

5. PASSAGE FROM THE \mathcal{A}_1 -EQUIVALENCE TO THE $\tilde{\mathcal{A}}$ -EQUIVALENCE

To get the normal forms of Theorem 2.1, in view of the result of Proposition 4.3, it suffices to show that the terms in $y(t)$ of a Puiseux parametrization of order k such that $k > \lambda$ and $k \in (\Lambda \setminus \Lambda^2) - v_0$, may be eliminated without changing the preceding terms.

Terms of order $k \in S - v_0$, where S is as in Proposition 4.2, excepting $k = \lambda$, may be eliminated by **EC2**. The only remaining possibility are terms of order $v_1 + \lambda - v_0$, when $g \geq 2$ and $n_1 = 2$ (cf. Proposition 4.2), which we will show below how to eliminate them without changing the preceding ones.

To do this, we will need to analyze more closely the $\tilde{\mathcal{A}}$ -action on Puiseux parametrizations.

Let $\varphi(t) = (t^{v_0}, y(t))$, where $y(t) = \sum_i a_i t^i$, and let σ and ρ as in (2.2), but with $r = 1$, $p = \beta Y + p_1$, where $\beta \in \mathbb{C}$ and $p_1, q \in \mathcal{M}_2^2$.

Now, considering the expression of ρ in (2.2), raising both sides to the power i and then applying the binomial expansion we get

$$t_1^i = t^i \left[\sum_{j=0}^{\infty} \binom{i/v_0}{j} \left(\frac{p(t)}{t^{v_0}} \right)^j \right],$$

where $p(t) = \varphi^*(p)$.

By using this in the expression $y(t) = \sum_i a_i t^i$, we get

$$y(t_1) = y(t) + \sum_i a_i t^i \frac{i}{v_0} \frac{p(t)}{t^{v_0}} + A(t),$$

where

$$A(t) = \sum_i a_i t^i \sum_{j=2}^{\infty} \binom{i/v_0}{j} \left(\frac{p(t)}{t^{v_0}} \right)^j.$$

Now, from the expression of $y_1(t_1)$ in (2.3) we get

$$(5.1) \quad y_1(t_1) = y(t_1) + B(t),$$

where, if we put $q(t) = \varphi^*(q)$,

$$(5.2) \quad B(t) = \frac{q(t)x'(t) - p(t)y'(t)}{x'(t)} - A(t).$$

Proposition 5.1. *Let $\varphi(t_1) = (t_1^{v_0}, y(t_1))$, where $y(t_1) = t_1^{v_1} + t_1^\lambda + \sum_{i>\lambda} a_i t_1^i$, be a Puiseux parametrization, such that the genus of φ is greater than 1, and $n_1 = 2$. Then there exists $y_1(t_1) = t_1^{v_1} + t_1^\lambda + \sum_{i>\lambda} a'_i t_1^i$, with $a'_i = a_i$, for $i < v_1 + \lambda - v_0$ and $a'_{v_1+\lambda-v_0} = 0$, such that $(t_1^{v_0}, y_1(t_1)) \sim_{\tilde{\mathcal{A}}} (t_1^{v_0}, y(t_1))$.*

PROOF: Let $\beta \in \mathbb{C}$ and $p_1, q \in \mathcal{M}_2^2$, as above. We will show that we may choose p_1 and q , in such a way that $\text{ord}_{t_1}(B(t)) = v_1 + \lambda - v_0$, where B is as in (5.2), and then, by adjusting the value of β , we may make this term cancel the corresponding one in $y(t_1)$ in equation (5.1).

Since $t_1 = \rho(t)$, with ρ an automorphism of \mathcal{O}_1 , we have that

$$\text{ord}_{t_1}(B(t)) = \text{ord}_t(B(t)),$$

so, we may work with the powers of t in the expression of $B(t)$.

Remark that $n_1 = 2$ implies that $m_1 v_0 = 2v_1$, where $m_1 = \beta_1/e_1 = v_1/e_1$.

Now we choose $p_1 = 0$ and $q = \frac{v_1}{v_0}\beta X^{m_1-1} + g$, with $g \in \mathcal{M}_2^2$ such that $v_\varphi(g) > (m_1 - 1)v_0$.

Let us write

$$B(t) = B_0(t) + B_1(t) + B_2(t),$$

where

$$B_0(t) = \beta \frac{(v_1/v_0)x(t)^{m_1-1}x'(t) - y(t)y'(t)}{x'(t)},$$

$$B_1(t) = \frac{g(t)x'(t)}{x'(t)} = g(t),$$

and

$$B_2(t) = -A(t) = -\sum_{i \geq v_1} a_i t^i \left(\sum_{j=2}^{\infty} \binom{i/v_0}{j} \left(\frac{\beta y(t)}{t^{v_0}} \right)^j \right).$$

A direct computation shows that, if $\beta \neq 0$, then

$$\begin{aligned} v_\varphi(B_0) &= v_\varphi((v_1/v_0)x(t)^{m_1-1}x'(t) - y(t)y'(t)) - (v_0 - 1) = \\ &= v_1 + \lambda - 1 - v_0 + 1 = v_1 + \lambda - v_0. \end{aligned}$$

On the other hand, by expanding $B_2(t)$ one sees that there will be terms either of degree greater than $v_1 + \lambda - v_0$, or of degree $rv_0 + sv_1$, greater than $(m_1 - 1)v_0$, which can be eliminated by a suitable choice of $g(t)$. \blacksquare

With this last proposition we finished the proof of the existence part of Theorem 2.1, concerning the normal forms.

Now, to prove that if two Puiseux parametrizations are \mathcal{A} -equivalent, then they are conjugate under homothety, it will be sufficient to prove that if two Puiseux parametrizations are $\tilde{\mathcal{A}}$ -equivalent, then they are equal, because the \mathcal{A} -action is decomposed into the $\tilde{\mathcal{A}}$ -action and the \mathcal{H} -action.

Fixing a set Λ of values of differentials in the equisingularity class determined by a semigroup Γ , let us consider the linear space

$$N_\Lambda = \{(t^{v_0}, t^{v_1} + t^\lambda + \sum_{j > \lambda} a_j t^j) \in \Sigma_\Gamma; a_j = 0, \text{ for } j \in \Lambda - v_0\}.$$

If we denote by N_Λ^k the space $j^k(N_\Lambda)$, we have the following lemma:

Lemma 5.2. *If $\alpha \in N_\Lambda$, then for all $k > \lambda$, we have*

$$N_\Lambda^k \cap T_{j^k(\alpha)}(\tilde{\mathcal{A}}^k \cdot j^k(\alpha)) = \{j^k(\alpha)\}.$$

PROOF: Suppose the assertion not true. Take k minimal with the following property:

$$N_\Lambda^k \cap T_{j^k(\alpha)}(\tilde{\mathcal{A}}^k \cdot j^k(\alpha)) \neq \{j^k(\alpha)\}.$$

So, there exists $\beta \in N_\Lambda^k \cap T_{j^k(\alpha)}(\tilde{\mathcal{A}}^k \cdot j^k(\alpha))$ such that $\beta \neq j^k(\alpha)$ and $j^{k-1}(\beta) = j^{k-1}(\alpha)$. Therefore, there exists $b \in \mathbb{C}^*$ such that

$$\beta - j^k(\alpha) = (0, bt^k) \in T_{j^k(\alpha)}(\tilde{\mathcal{A}}^k \cdot j^k(\alpha)).$$

Hence, from Proposition 3.3, it follows that $k \in \Lambda - v_0$. But, since $j^k(\alpha) \in N_\Lambda^k$, it follows that $j^k(\alpha) = j^{k-1}(\alpha)$. So, for some $b \neq 0$,

$$\beta = j^{k-1}(\alpha) + (0, bt^k).$$

But, since $\beta \in N_\Lambda^k$, one should have $b = 0$, which is a contradiction. \blacksquare

Now we proceed to prove the uniqueness of the $\tilde{\mathcal{A}}$ -normal forms.

Let $\varphi(t) = (t^{v_0}, t^{v_1} + t^\lambda + \sum_{j>\lambda} a_j t^j) \in \Sigma_\Gamma$ be a Puiseux parametrization with $\Lambda_\varphi = \Lambda$. We denote by $\tilde{\mathcal{A}}^{c-1} \cdot \varphi$ the orbit of φ in Σ_Γ , with respect to the $\tilde{\mathcal{A}}^{c-1}$ -action.

We want to show that

$$N_\Lambda \cap \tilde{\mathcal{A}}^{c-1} \cdot \varphi = \{\varphi\}.$$

Indeed, if $N_\Lambda \cap \tilde{\mathcal{A}}^{c-1} \cdot \varphi \neq \{\varphi\}$, take $\varphi_1 \in N_\Lambda \cap \tilde{\mathcal{A}}^{c-1} \cdot \varphi$, with $\varphi_1 \neq \varphi$. Since $\tilde{\mathcal{A}}^{c-1} \cdot \varphi$ is arcwise connected, there exists an arc in $\tilde{\mathcal{A}}^{c-1} \cdot \varphi$ joining φ to φ_1 . Since reduction to the normal form is continuous, it follows that φ wouldn't be an isolated point in $N_\Lambda \cap \tilde{\mathcal{A}}^{c-1} \cdot \varphi$. But this is a contradiction because of Lemma 5.2.

6. ZARISKI'S PROBLEM AND COMPUTATIONAL ASPECTS

Our methods, more than describing all normal forms (up to homotheties) of plane branches, with respect to analytic equivalence, give an effective way to obtain the normal form of a given branch and, as well, to distinguish from analytic point of view two given plane branches.

Indeed, since the set Λ is an analytic invariant, it is the same for the branch and its normal form, so given two parametrizations φ_1 and φ_2 with same semigroup of values, we compute with the procedures exposed in [9] (specially Algorithm 4.10) the sets Λ for both. If these are distinct, the branches are not equivalent. If they are equal, we proceed to put the parametrizations under they normal forms. To put a given parametrization φ into its normal form, it is enough to consider changes of coordinates corresponding to an element of the group $\tilde{\mathcal{A}}^{l-v_0}$, where l is the greatest

integer not in Λ . More precisely, taking

$$p = \sum_i \alpha_i \prod_{j=0}^g h_j^{a_{ij}}, \quad q = \sum_i \gamma_i \prod_{j=0}^g h_j^{b_{ij}},$$

where h_0, h_1, \dots, h_g are elements of a standard basis for the local ring \mathcal{O}_f of the branch (f) associated to φ (see [9] for the definition (Definition 2.1) and how to compute them (Algorithm 3.2)), and the α_i 's and γ_i 's are parameters, such that in the development in power series the smallest order term of p is greater than v_0 and of q is greater than v_1 and the terms of orders belonging to Γ in p (resp. q) are less than $l - v_1$ (resp. less than $l - v_0$).

Performing an action as (2.2) with the p and q as above, which is computationally possible, we impose conditions on the coefficients α_i and γ_i in order to bring the parametrization into its normal form, in the way we did during the proof of Proposition 5.1.

This done, the analytic equivalence reduces to verify homothety, which is trivial.

Let us remark that Ebey (cf. [4], Theorem 5), by using arguments from the theory of algebraic groups, predicted the existence of some kind of normal forms under the \mathcal{A}_1 -equivalence (cf. our Proposition 4.3) which he called *canonical forms*, but without any indication on how they could be obtained nor how they should look like.

The *stratified moduli problem* is also solved, since it is the disjoint union of quotients of a finite number of semi-algebraic sets, by finite groups, corresponding to the quotients modulo finitely many homotheties of the normal form corresponding to a given Λ under the $\tilde{\mathcal{A}}$ -equivalence. The sets Λ and the conditions on the coefficients that determine them, for a fixed equisingularity class, may be computed by the algorithms we developed in [9]. One of these set Λ_{gen} corresponds to the generic branch, easily recognized by the open conditions on the coefficients.

Finally, the dimension of the component of the moduli corresponding to a given set Λ is determined by the normal form and is at most equal to the number of gaps of Λ greater than λ , since some of the coefficients of the parametrization may be fixed constants. The dimension of the generic component is exactly equal to the number of gaps of Λ_{gen} greater than λ , since in this case, no coefficient in the corresponding normal form is constant.

7. SOME EXAMPLES

In what follows we give two concrete examples of the application of our method. The first example will describe a result obtained in [8], and the second one is a new example which we relate to a question posed by Zariski in [11].

Example 7.1. *The table below gives the analytic classification of plane branches in the equisingularity class of $\Gamma = \langle 6, 9, 19 \rangle$.*

Condition	Normal Form
$b \notin \{-\frac{1}{2}, \frac{29}{18}\}$	$(t^6, t^9 + t^{10} + bt^{11} + b_1t^{14} + b_2t^{17})$
$b = \frac{29}{18}$	$(t^6, t^9 + t^{10} + bt^{11} + b_1t^{14} + b_2t^{17} + b_3t^{23})$
$b = -\frac{1}{2}$ $A \neq 0$	$(t^6, t^9 + t^{10} + bt^{11} + b_1t^{14} + b_2t^{17} + b_3t^{20})$
$b = -\frac{1}{2}$ $A = 0$	$(t^6, t^9 + t^{10} + bt^{11} + b_1t^{14} + b_2t^{17} + b_3t^{20} + b_4t^{26})$

Where $A = 14 + \frac{769}{2}b_1 - 532b_2 - 576b_1^2$. Moreover, two branches belonging to the same normal form are equivalent if, and only if, they are equal.

In the above example we have that the stratified moduli space consists of four strata, all of them of dimension 3, with the first one corresponding to the generic stratum.

Example 7.2. Our second example deals with the classification of the equisingularity class given by the semigroup of values $\Gamma = \langle 7, 8 \rangle$.

Since the conductor of Γ is 42, we have

$$\Sigma_\Gamma = \{(t^7, t^8 + \sum_{8 < i < 42} a_i t^i); a_i \in \mathbb{C}\}.$$

Algorithm 4.10 of [9] and Theorem 2.1 give the following table:

Condition	Normal Form	$\Lambda \setminus \Gamma$
$a_{12} \neq \frac{13+9a_{11}^2}{8}$	$(t^7, t^8 + t^{10} + a_{11}t^{11} + a_{12}t^{12} + a_{13}t^{13} + a_{20}t^{20})$	17, 25, 26 33, 34, 41
$a_{13} \neq \frac{39}{10}a_{11} + \frac{27}{20}a_{11}^3$	$(t^7, t^8 + t^{10} + a_{11}t^{11} + \frac{13+9a_{11}^2}{8}t^{12} + a_{13}t^{13} + a_{19}t^{19})$	17, 25, 27 33, 34, 41
$a_{20} \neq B$	$(t^7, t^8 + t^{10} + a_{11}t^{11} + \frac{13+9a_{11}^2}{8}t^{12} + (\frac{39}{10}a_{11} + \frac{27}{20}a_{11}^3)t^{13} + a_{19}t^{19} + a_{20}t^{20})$	17, 25, 33 34, 41
$a_{20} = B$	$(t^7, t^8 + t^{10} + a_{11}t^{11} + \frac{13+9a_{11}^2}{8}t^{12} + (\frac{39}{10}a_{11} + \frac{27}{20}a_{11}^3)t^{13} + a_{19}t^{19} + a_{20}t^{20} + a_{27}t^{27})$	17, 25 33, 41
	$(t^7, t^8 + t^{11} + a_{12}t^{12} + a_{13}t^{13} + a_{20}t^{20})$	18, 25, 26 33, 34, 41
	$(t^7, t^8 + t^{12} + a_{13}t^{13} + a_{18}t^{18})$	19, 26, 27 33, 34, 41
$a_{18} \neq -\frac{1}{2}$	$(t^7, t^8 + t^{13} + a_{18}t^{18} + a_{19}t^{19} + a_{26}t^{26})$	20, 27, 33 34, 41
	$(t^7, t^8 + t^{13} - \frac{1}{2}t^{18} + a_{19}t^{19} + a_{26}t^{26})$	20, 27 34, 41
$a_{20} \neq \frac{121}{120}a_{19}^2$	$(t^7, t^8 + t^{18} + a_{19}t^{19} + a_{20}t^{20} + a_{27}t^{27})$	25, 33 34, 41
	$(t^7, t^8 + t^{18} + a_{19}t^{19} + \frac{121}{120}a_{19}^2t^{20} + a_{27}t^{27})$	25, 33, 41
	$(t^7, t^8 + t^{19} + a_{20}t^{20})$	26, 33 34, 41
	$(t^7, t^8 + t^{20} + a_{26}t^{26})$	27, 34, 41
	$(t^7, t^8 + t^{26} + a_{27}t^{27})$	33, 41
	$(t^7, t^8 + t^{27})$	34, 41
	$(t^7, t^8 + t^{34})$	41
	(t^7, t^8)	\emptyset

where

$$B = \frac{11}{4}a_{11}a_{19} - \frac{357}{512} - \frac{47399}{2560}a_{11}^2 - \frac{10097}{320}a_{11}^4 - \frac{17523}{1280}a_{11}^6 - \frac{2187}{1280}a_{11}^8.$$

Two parametrizations of the above table on the same line are equivalent if and only if they are homothetic, with respect to the appropriate root of unity $((\lambda - 8)$ -th root of unity).

From the above table we see that the generic component of the moduli, corresponding to parametrizations on the first line, has dimension 4. There are six strata of dimension 3, three strata of dimension 2, three strata of dimension 1 and three strata of dimension 0.

Zariski dedicated Sections 4, 5 and 6 of Chapter VI, in [11], to the study of branches with semigroups of the form $\Gamma = \langle v_0, v_0 + 1 \rangle$, where the following result is proved:

THEOREM ([11], Théorème 6.12) *Let $v_0 \geq 5$, and for all $s \in \{2, \dots, v_0 - 2\}$, define*

$$\mathcal{L}_s = \{sv_0 + s + 2, sv_0 + s + 3, \dots, sv_0 + s + v_0 - 1 - s\}.$$

Let

$$\varphi = (t^{v_0}, t^{v_0+1} + a_{v_0+3}t^{v_0+3} + \dots + a_{2v_0-1}t^{2v_0-1} + \sum_{i \in \bigcup_{s=2}^q \mathcal{L}_s} a_i t^i),$$

where $q = \lfloor \frac{v_0-3}{2} \rfloor$ and $a_i = 0$, whenever i is one of the first $s + 1$ elements of \mathcal{L}_s , for all $2 \leq s \leq q$.

Then two generic parametrizations of the above form are \mathcal{A} -equivalent if, and only if, they are homothetic.

In [11], Zariski remarks that the above theorem is true for $2 \leq v_0 \leq 6$ without the condition on the genericity of the parameters (Remarque 6.14), and asks the following question:

Is the above theorem true without the assumption of the genericity on the coefficients of the parametrizations?

The answer is no! And an example may be given considering branches with semigroup $\Gamma = \langle 7, 8 \rangle$.

Consider

$$\varphi = (t^7, t^8 + t^{10} + t^{11} + \frac{11}{4}t^{12} + a_{13}t^{13} + a_{20}t^{20}),$$

with $a_{13} \neq \frac{21}{4}$. Obviously, φ is in the form of the above theorem.

If we consider changes of coordinates as in (2.2) with

$$p = b_1x^2 - \frac{3}{2}b_1xy - \frac{1}{4}b_1y^2 + b_2x^3 + b_3x^2y + b_4xy^2 + b_5y^3$$

and

$$q = \frac{8}{7}b_1xy - \frac{12}{7}b_1y^2 + \left(-\frac{135}{28}b_1 - \frac{15}{14}a_{13}b_1\right)x^3 + b_6x^2y + b_7xy^2 + b_8y^3,$$

where

$$\begin{aligned}
b_2 &= \frac{4}{7}b_1^2 - \frac{227}{6}b_1 + \frac{199}{24}a_{13}b_1 - \frac{2}{3}b_3 - \frac{8}{3}b_5, \\
b_3 &= 6b_4 - \frac{45}{2}b_1a_{13}^2 + \frac{9565}{16}b_1 - 40b_8 + \frac{72}{7}b_1^2 + \frac{4297}{16}a_{13}b_1, \\
b_4 &= (4a_{13} - 21)^{-1} \frac{1}{1120} (-2720a_{13}b_1^2 + 557690a_{13}b_1 - \\
&\quad 277200b_1a_{13}^2 + 16800b_1a_{13}^3 + 26880a_{13}b_5 + \\
&\quad 2459289b_1 - 141120b_5 + 14280b_1^2 + 2688b_1a_{20}), \\
b_6 &= -\frac{25}{4}b_1 - \frac{29}{28}a_{13}b_1 + \frac{8}{7}b_2 + \frac{4}{49}b_1^2, \\
b_7 &= -\frac{3}{2}a_{13}b_1 + \frac{8}{7}b_3 - \frac{641}{56}b_1 - \frac{12}{49}b_1^2, \\
b_8 &= -\frac{52}{7}b_1 + \frac{8}{7}b_4 - \frac{81}{28}a_{13}b_1 + \frac{8}{7}b_2 + \frac{18}{49}b_1^2,
\end{aligned}$$

we have

$$\sigma \circ \varphi \circ \rho^{-1}(t_1) = (t_1^7, t_1^8 + t_1^{10} + t_1^{11} + \frac{11}{4}t_1^{12} + a_{13}t_1^{13} + (a_{20} + 5b_1(\frac{3}{4} - \frac{1}{7}a_{13}))t_1^{20}).$$

By choosing conveniently b_1 we see that the normal form of φ is given in the second row of the table in Example 7.2, but φ itself is not in normal form (this gives the reduction of φ to normal form).

Since $a_{13} \neq \frac{21}{4}$, then for each $b_1 \neq 0$ we get a parametrization \mathcal{A} -equivalent to φ , as described in Zariski's Theorem, without being homothetically equivalent to φ , giving a negative answer to Zariski's question.

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Abramo Hefez
Universidade Federal Fluminense
Instituto de Matemática
R. Mario Santos Braga, s/n
24020-140 Niterói, RJ - Brazil
E-mail: hefez@mat.uff.br

Marcelo E. Hernandez
Universidade Estadual de Maringá
Departamento de Matemática
Av. Colombo, 5790
87020-020 Maringá, PR - Brazil
E-mail: mehernandes@uem.br